HEAT TRANSFER IN TURBULENT NON-NEWTONIAN FLOW*

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The effect of non-Newtonian Prandtl number on the distribution of resistance to heat transfer is examined. Two new analogies between heat and momentum transfer are developed, one of which is shown, by comparison with experiment, to be superior to another more complex and recently published analogy-type expression. It is concluded that more accurate information is needed on conditions near the wall before reliable characterization of heat transfer can be made in such systems.

The complexities of rigorous analysis of this subject have so far proved prohibitive, but useful results have been obtained on the basis of analogies between the processes of heat and momentum transfer, as first proposed by Osborne Reynolds in 1874. Many of the refinements in the theory of analogy for Newtonian materials since that time are reviewed by Knudsen and Katz [1]. Attention will here be mainly confined to two new analogies and one recent analogy between heat and momentum transfer for non-Newtonian fluids in tubes.

Analyses by Metzner and Friend [2, 3, 4] refine and extend an earlier development by Reichardt which defined the general form of a solution allowing for slight but unspecified turbulence close to the wall. The purely viscous non-Newtonian expression [4] was restricted to conditions of effectively isothermal heat transfer and, according to Petersen and Christiansen [5], yields predictions which deviate increasingly from experimental values with increasingly non-Newtonian behavior. The precedure presented by Petersen and Christiansen [5], however, is highly complex both in formulation and application.

The next section explains why assumptions regarding flow conditions in the vicinity of the wall are of crucial importance to the success or otherwise of heat-transfer relationships based on analogies in the case of most non-Newtonian materials. Emphasis will be on power-law non-Newtonian substances for which, in tube flow

$$\tau_{rx} = K\left(-\frac{du}{dr}\right)^n. \tag{1}$$

The general rheology and characterization of these and other non-Newtonian fluids has been described by Skelland [6].

The Distribution of Resistance to Heat Transfer in Turbulent Flow

The heat-transfer coefficient for a power-law fluid in turbulent flow through a round tube will be a function of the variables listed below (symbols are defined at the end of the article):

$$h = f(D,\ V,\ \rho,\ c_p,\ k,\ K,\ n). \tag{2}$$

Dimensional analysis gives the following result for a given n:

$$\frac{hD}{k} = f_1\left[\frac{D^n V^{2-n}\rho}{K},\ \frac{c_p K}{k}\left(\frac{V}{D}\right)^{n-1}\right] \tag{3}$$

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or

$$\mathrm{Nu} = f(\mathrm{Re}^+, \ \mathrm{Pr}^+). \tag{4}$$

The radial heat flux at y may be written as

$$q = -(\alpha_t + \varepsilon_H)\,\frac{d(c_p\rho T)}{dy}, \tag{5}$$

where ε_H is an eddy thermal diffusivity. For regions of small y near the wall $q \cong q_w$, and

$$q_w \frac{\rho}{K}\left(\frac{D}{V}\right)^{n-1} = -\left(\frac{k}{\rho c_p} + \varepsilon_H\right)\frac{\rho}{K}\left(\frac{D}{V}\right)^{n-1}\frac{d(c_p\rho T)}{dy}$$

or

$$1 = -\left[\frac{1}{\mathrm{Pr}^+} + \frac{\varepsilon_H}{(K/\rho)(V/D)^{n-1}}\right]\frac{K}{\rho q_w}\left(\frac{V}{D}\right)^{n-1} c_p\rho\,\frac{dT}{dy}. \tag{6}$$

A dimensionless temperature is commonly defined as

$$T^+ = \frac{(T_w - T)\,\rho c_p u^*}{q_w}\ ; \quad \therefore dT = -\frac{q_w dT^+}{\rho c_p u^*}\ , \tag{7}$$

and if the dimensionless distance from the wall is given by

$$y^+ = \frac{y^n (u^*)^{2-n}\rho}{K}\ , \tag{8}$$

where u* is the friction velocity, $\sqrt{\tau_w g_c/\rho}$, then

$$dy = \frac{K dy^+}{n y^{n-1}(u^*)^{2-n}\rho}\ . \tag{9}$$

Combination of Eqs. (6), (7), and (9) yields the dimensionless expression

$$1 = \left[\frac{1}{\mathrm{Pr}^+} + \frac{\varepsilon_H}{(K/\rho)(V/D)^{n-1}}\right] n\left(\frac{y}{D}\cdot\frac{V}{u^*}\right)^{n-1}\frac{dT^+}{dy^+}\ . \tag{10}$$

Consider now some particular point close enough to the wall for ε_H to be effectively zero for the case of two fluid systems with Prandtl numbers $N^+_{\mathrm{Pr}_1}$ and $N^+_{\mathrm{Pr}_2}$. Suppose this difference in N^+_{Pr} is due solely to differences in consistency index, K, the quantities D, V, y, n, c_p, k, and ρ/K having respectively the same values in the two systems, so that $N^+_{\mathrm{Re}_1} = N^+_{\mathrm{Re}_2}$. Equation (10) may be written for each system, and taking ratios,

$$1 = \frac{\mathrm{Pr}_2^+}{\mathrm{Pr}_1^+}\left(\frac{u_2^*}{u_1^*}\right)^{n-1}\frac{(dT^+/dy^+)_1}{(dT^+/dy^+)_2}\ . \tag{11}$$

An approximate relation between Fanning friction factor and Reynolds number in turbulent flow is given by Dodge and Metzner [7]; it takes the following form for power-law fluids:

$$f = \frac{\tau_w}{\rho V^2/2} = \frac{\alpha}{(C_n \mathrm{Re}^+)^\beta}\ ; \quad 5\cdot 10^3 \leqslant C_n\,\mathrm{Re}^+ \leqslant 10^5, \tag{12}$$

where α and β are functions of n only [6, 7]. The definitions of u* and f show that $u^* = V\sqrt{f/2}$; but n, V, and N^+_{Re} have respectively the same values in the two systems, so that $u_1^* = u_2^*$. Equation (11) accordingly becomes

$$\frac{(dT^+/dy^+)_2}{(dT^+/dy^+)_1} = \frac{\mathrm{Pr}_2^+}{\mathrm{Pr}_1^+}\ . \tag{13}$$

This shows that, for given values of Reynolds number and n, increasing the non-Newtonian Prandtl number leads to an equivalent increase in the dimensionless temperature gradient in the wall region (where $\varepsilon_H \to 0$, $q \to q_w$). This, of course, corresponds to locating more and more of the major resistance to heat transfer within the sublayers near the wall with increasing N^+_{Pr}. Non-Newtonian Prandtl numbers are in fact high, so that assumptions concerning ε_H near the wall, the nature and thickness of the laminar sublayer, and other quantities in the wall region become of critical importance in the analysis of turbulent non-Newtonian heat transfer. Relationships based upon several different assumptions about conditions in the vicinity of the wall will now be considered.

Analogy Assuming a Laminar Sublayer and a Turbulent Core

A Taylor–Prandtl type of analysis will here be extended to non-Newtonian fluids for which the shearing behavior is described by the power law. The laminar sublayer adjacent to the wall has thickness δ_L and the temperature and velocity in the x direction at $y = \delta_L$ are T_L and u_L.

Consider first the transfer of momentum and heat in the laminar sublayer, which is supposed to be thin enough to assume linear distributions of velocity and temperature.

$$\tau_w = K\left(\frac{du}{dy}\right)^n = K\left(\frac{u}{y}\right)^n = K\left(\frac{u_L}{\delta_L}\right)^n, \quad y \leqslant \delta_L; \tag{14}$$

$$q_w = -k\frac{dT}{dy} = -k\,\frac{(T_L - T_w)}{\delta_L}, \quad y \leqslant \delta_L. \tag{15}$$

from which

$$\frac{q_w}{\tau_w} = -\frac{k(T_L - T_w)}{\delta_L^{1-n} u_L^n K}. \tag{16}$$

Transfer in the turbulent core ($y > \delta_L$) is described by Reynolds' analogy, in which aggregates of fluid travel, on the average, back and forth between the edge of the laminar sublayer ($u = u_L$, $T = T_L$) and locations where velocity and temperature have bulk average values ($u = V$, $T = T_m$). These fluid aggregates carry the momentum and temperature corresponding to the location at which the aggregates first attain identity. Thus for a constant-property fluid and an average aggregate mass m: momentum transfer towards the sublayer = $m(V - u_L)$; heat transfer away from the sublayer = $-mc_p(T_m - T_L)$ or

$$\frac{q}{\tau_w} = \frac{-c_p(T_m - T_L)}{V - u_L}. \tag{17}$$

Equations (16) and (17) both apply at $y = \delta_L$. Solving Eq. (16) for $(T_w - T_L)$ and Eq. (17) for $(T_L - T_m)$ and adding the results,

$$c_p(T_w - T_m) = \frac{q_w}{\tau_w}\left[\frac{\delta_L^{1-n} u_L^n c_p K}{k} + V - u_L\right]$$

or

$$N_{\mathrm{St}} = \frac{q_w}{\rho V c_p (T_w - T_m)} = \frac{f/2}{1 + \dfrac{\delta_L^{1-n} u_L^n c_p K}{Vk} - \dfrac{u_L}{V}}. \tag{18}$$

Equation (18) is readily rearranged to

$$\mathrm{St} = \frac{f/2}{1 + \dfrac{u_L}{V}\left[\left(\dfrac{u_L}{V}\cdot\dfrac{D}{\delta_L}\right)^{n-1}\mathrm{Pr}^+ - 1\right]}. \tag{19}$$

Using Clapp's (8) assumption that at the edge of the laminar sublayer

$$u^+ = u_L/u^* = (y_L^+)^{1/n} = 5;$$

then since $u^* = V\sqrt{f/2}$

$$u_L = 5V\sqrt{f/2}. \tag{20}$$

and from Eq. (8) for $y = \delta_L$

$$\delta_L = 5\left(\frac{K}{(V\sqrt{f/2})^{2-n}\rho}\right)^{1/n}. \tag{21}$$

Combination of Eqs. (19) to (21) gives

$$\mathrm{St} = \frac{\mathrm{Nu}}{\mathrm{Re}^+\mathrm{Pr}^+} = \frac{f/2}{1 + 5\sqrt{\dfrac{f}{2}}\left[\left(\mathrm{Re}^+ + \dfrac{f}{2}\right)^{\frac{n-1}{n}}\mathrm{Pr}^+ - 1\right]}, \tag{22}$$

where N_{Nu}, N_{Re}^+, N_{Pr}^+, and f are defined by Eqs. (3), (4), and (12). The friction factor f is obtainable from Eq. (12). (Other means for evaluating f when $C_n N_{\mathrm{Re}}^+ > 10^5$ are given by Skelland [6].)

Alternative Analogy Assuming a Laminar Sublayer and a Turbulent Core

The quantities u_L/V and D/δ_L in Eq. (19) may be estimated in a different way which does not require Clapp's assumption concerning the thickness of the laminar sublayer, namely $y_L^+ = 5^n$. This involves derivation of an expression for turbulent velocity distribution in the tube as follows. Equation (12) may be expanded to give

$$\tau_w = \left(\frac{\alpha}{2^{\beta n+1}}\right) \rho V^{2-\beta(2-n)} \left(\frac{\gamma_1}{\rho}\right) R^{-\beta n} = \rho (u^*)^2,$$

where

$$\gamma_1 = 8^{n-1} K \left(\frac{3n+1}{4n}\right)^n = K/C_n, \tag{23}$$

Rearranging and eliminating the mean velocity V with the aid of the maximum velocity by putting $V = 0.817 \cdot u_m$ (Bogue and Metzner [9]),

$$\frac{u_m}{u^*} = \frac{1}{0.817} \left(\frac{2^{\beta n+1}}{\alpha}\right)^{\frac{1}{2-\beta(2-n)}} \left[\frac{R^n (u^*)^{2-n} \rho}{\gamma_1}\right]^{\frac{\beta}{2-\beta(2-n)}}. \tag{24}$$

Next, to quote Schlichting [10, line 20], "It is now natural to assume that this equation is valid for any wall distance y, and not only for the pipe axis (wall distance y = R). Hence we obtain from Eq. (24)"

$$\frac{u}{u^*} = \frac{1}{0.817} \left(\frac{2^{\beta n+1}}{\alpha}\right)^{\frac{1}{2-\beta(2-n)}} \left[\frac{y^n (u^*)^{2-n} \rho}{\gamma_1}\right]^{\frac{\beta}{2-\beta(2-n)}}. \tag{25}$$

and taking the ratio of Eqs. (25) to (24),

$$\frac{u}{u_m} = \left(\frac{y}{R}\right)^{\frac{\beta n}{2-\beta(2-n)}}. \tag{26}$$

This expression was first derived by Skelland [11] using a different method, and was shown to be in good agreement with experiment. For Newtonian fluids $n = 1.0$, $\beta = 0.25$, and Eq. (26) reduces to the well known Prandtl one-seventh power law.

Equation (26) is manipulated to give the following, after setting $R = D/2$, $u_m = V/0.817$, $u = u_L$, $y = \delta_L$:

$$\frac{D}{\delta_L} = 2 \left(\frac{0.817 u_L}{V}\right)^{\frac{\beta(2-n)-2}{\beta n}}. \tag{27}$$

Application of Eq. (12) to the laminar sublayer yields

$$\begin{aligned} \tau_w &= \frac{\alpha}{2} \rho V^2 (C_n \mathrm{Re}^+)^{-\beta} = K \left(\frac{u_L}{\delta_L}\right)^n, \\ u_L^n &= \frac{\alpha}{2K} \rho V^2 \frac{\gamma_1}{D^n V^{2-n} \rho} (C_n \mathrm{Re}^+)^{1-\beta} \delta_L^n, \\ \frac{\delta_L}{D} &= \left(\frac{2K}{\alpha \gamma_1}\right)^{\frac{1}{n}} \left(\frac{u_L}{V}\right) (C_n \mathrm{Re}^+)^{\frac{\beta-1}{n}}. \end{aligned} \tag{28}$$

Combining Eqs. (23), (27), and (28) and solving for u_L/V

$$\frac{u_L}{V} = \left[2^{\beta(n+1)} 0.817^{\beta(2-n)-2} \left(\frac{C_n}{\alpha}\right)^{\beta}\right]^{\frac{1}{2(1-\beta)}} (C_n \mathrm{Re}^+)^{-\frac{\beta}{2}}. \tag{29}$$

Equation (27) is inserted in Eq. (19) to obtain

$$\mathrm{St} = \frac{f/2}{1 + \frac{u_L}{V} \left\{ \left[2 (0.817)^{\frac{\beta(2-n)-2}{\beta n}} \left(\frac{u_L}{V}\right)^{\frac{2(\beta-1)}{\beta n}} \right]^{n-1} \mathrm{Pr}^+ - 1 \right\}}, \tag{30}$$

where u_L/V is given by Eq. (29).

TABLE 1. Comparison between Experimental N_{Nu} and Values Calculated from Equations 22, 30, and 40 for Tubes

$c_n \mathrm{Re}^+$	$\frac{\mathrm{Pr}^+}{c_n}$	n	Nusselt Number: experimental (Clapp, [8])	equation 22	Clapp-Martinelli, Eq. 40; $\Omega' = 1.0$	equation 30
5,145	98,3	0,733	119,4	93,3	86,8	34,1
9,340	101,6	0,698	213,5	203,0	172,5	76,7
12,070	95,0	0,719	272,0	249,0	218,0	91,7
14,770	69,6	0,744	286,8	286,0	236,2	107,0
23,000	61,3	0,786	431,5	407,0	345,8	158,4

Analogy Assuming a Laminar Sublayer, a Buffer Layer, and a Turbulent Core

An extension of Martinelli's analogy was performed by Clapp [8] for power-law non-Newtonian fluids using the following Martinelli assumptions:

$$\frac{\tau}{\rho} = \frac{\tau_w}{\rho}\left(1 - \frac{y}{R}\right) = \frac{K}{\rho}\left(\frac{du}{dy}\right)^n + \varepsilon_M \frac{du}{dy}, \tag{31}$$

$$\frac{q}{\rho c_p} = \frac{q_w}{\rho c_p}\left(1 - \frac{y}{R}\right) = -\left(\frac{k}{\rho c_p} + \varepsilon_H\right)\frac{dT}{dy}, \tag{32}$$

where the eddy diffusivities of heat and momentum are related by

$$\varepsilon_H = \Omega' \varepsilon_M . \tag{33}$$

Clapp [8] developed the following expressions for velocity distribution in the tube.

Laminar sublayer:

$$u^+ = (y^+)^{1/n}, \quad 0 \leqslant y^+ \leqslant 5^n . \tag{34}$$

Buffer layer:

$$u^+ = \frac{5.0}{n} \ln y^+ - 3.05, \quad 5^n \leqslant y^+ \leqslant y_2^+ . \tag{35}$$

Turbulent core:

$$u^+ = \frac{2.78}{n} \ln y^+ + \frac{3.8}{n}, \quad y^+ > y_2^+ . \tag{36}$$

The laminar sublayer was assumed to extend to $y^+ = 5^n$; y_2^+ is located at the intersection of Eqs. (35) and (36). These relationships were used to obtain the temperature profile in the tube, as shown below.

Laminar sublayer:

$$T_w - T = Q_w \frac{\tau_w c_p \Omega'}{k}\left(\frac{\tau_w}{K y^+}\right)^{-1/n}, \quad 0 > y^+ < 5^n. \tag{37}$$

Buffer layer:

$$T_L - T = Q_w \int_{y_L^+}^{y^+} \left[\frac{nk}{\tau_w c_p \Omega'}\left(\frac{\tau_w}{K}\right)^{1/n} (y^+)^{\frac{n-1}{n}} + \frac{n y^+}{5} - n(5)^{n-1}\right]^{-1} dy^+, \quad 5^n < y^+ < y_2^+ . \tag{38}$$

Turbulent core:

$$T_2 - T = Q_w \frac{G}{n} \ln\left[\frac{c_n \mathrm{Re}^+ (\sqrt{f/2})^{2-n}}{8 y_2^+} 4^n \left(\frac{y}{R}\right)^n\right], \quad y^+ > y_2^+ , \tag{39}$$

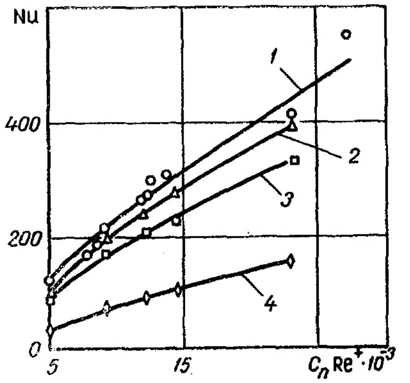

Fig. 1

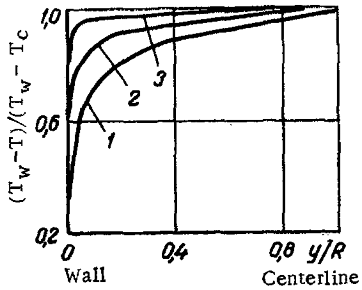

Fig. 2

Fig. 1. Comparison between experimental Nusselt numbers and values calculated from Eqs. (22), (30), and (40) for tubes. ($0.69 < n < 0.79$; $61 < \mathrm{Pr}^+/C_n < 104$): 1) experimental; 2) Eq. (22); 3) Eq. (40); 4) Eq. (30).

Fig. 2. Radial temperature distribution in a tube for turbulent flow of power-law non-Newtonian fluids (Eqs. (37), (38), and (39)). $N_{\mathrm{Re}^+} = 10^4$; $n = 0.5$; $\Omega' = 1.0$: 1) $N_{\mathrm{Pr}^+} = 1$; 2) $N_{\mathrm{Pr}^+} = 10$; 3) $N_{\mathrm{Pr}^+} = 100$.

where $Q_w = q_w/\Omega' c_p \rho u^*$; $G = 2.78$. These equations reduce to those derived by Martinelli for high-Prandtl-number Newtonian fluids when $n = 1.0$.

The corresponding Stanton number is

$$\mathrm{St} = \frac{\Omega' \sqrt{f/2}}{F_a + F_b + F_c}\left(\frac{T_w - T_c}{T_w - T_m}\right), \tag{40}$$

where

$$F_a = \frac{\tau_w c_p \Omega'}{k}\left(\frac{\tau_w}{K y^+}\right)^{-1/n};$$

$$F_b = \int_{y_L^+}^{y_2^+}\left[\frac{nk}{\tau_w c_p \Omega'}\left(\frac{\tau_w}{K}\right)^{1/n}(y^+)^{\frac{n-1}{n}} + \frac{ny^+}{5} - n(5)^{n-1}\right]^{-1} dy^+;$$

$$F_c = \frac{G}{n}\ln C_n \mathrm{Re}^+ (\sqrt{f/2})^{2-n}\frac{4^n}{8y_2^+}.$$

Comparisons between Nusselt numbers calculated from Eqs. (22), (30), and (40) and experimental values appear in Table 1. (Calculations are by slide rule and therefore approximate.) Results are also plotted in Fig. 1, which contains six additional experimental points from Clapp [8].

It is interesting to note that the least sophisticated analysis culminating in Eq. (22) gives results which, although low, are substantially closer to the experimental measurements than those obtained either by the more complicated Eq. (30) or by the still more complex Martinelli-type treatment leading to Eq. (40).

This is consistent with current postulates which replace the hypothetical laminar sublayer with some degree of eddying motion which does not decay to zero until the solid surface is reached. Thus Eq. (22) allows full turbulence beyond the laminar sublayer [$y^+ > (y_L^+ = 5^n)$], whereas Eq. (40) allows only reduced turbulence in the buffer layer ($5^n < y^+ < y_2^+$). Equation (30), although allowing full turbulence beyond the laminar sublayer, nevertheless requires a laminar sublayer more than twice as thick as either of the other two models. This may be seen from consideration of the definitions of y^+, N_{Re}^+, and u^*, which shows that

$$y_L^+ = \left(\frac{D}{\delta_L}\right)^{-n}\left(\frac{f}{2}\right)^{\frac{2-n}{2}}\mathrm{Re}^+. \tag{41}$$

TABLE 2. Comparison between Laminar Sublayer Thicknesses (y_L^+) Corresponding to Equations (22), (30), and (40)

$C_n Re^+$	5,145	9,340	12,070	14,770	23,000
y_L^+ corresponding to Eqs. (22) and (40)	3,25	3,07	3,17	3,31	3,55
y_L^+ corresponding to Eq. (30)	7,1	6,37	6,7	7,34	8,0

In the case of the alternative analogy leading to Eq. (30), D/δ_L is obtained from Eqs. (27) and (29). Dimensionless thicknesses of the laminar sublayer y_L^+, calculated from Eq. (41) and corresponding to Eq. (30), and from $y_L^+ = 5^n$ corresponding to Eqs. (22) and (40), are compared in Table 2.

Table 1, then, shows improvement in prediction with increasing allowance for turbulence near the wall. If, however, we assume full turbulence right up to the solid surface, Eq. (17) is written with (T_W, 0) in place of (T_L, u_L) to give $N_{St} = f/2$. This, unfortunately, yields N_{Nu} values between 8 and 16 times higher than the experimental results in Table 1.

Further information on this problem is provided by Fig. 2, where Eqs. (37), (38), and (39) have been used to plot the radial temperature distribution in dimensionless form as $(T_W - T)/(T_W - T_c)$ versus y/R with N_{Pr}^+ as parameter. The plots are for $N_{Re}^+ = 10{,}000$, $\Omega' = 1.0$, and $n = 1/2$ and were obtained after utilizing the following relationships

$$\frac{\tau_w c_p}{k}\left(\frac{\tau_w}{K}\right)^{-1/n} = \left(\frac{6n+2}{n}\right)^{n-1} \mathrm{Pr}^+, \tag{42}$$

$$y^+ = \frac{\mathrm{Re}^+}{2^n}(\sqrt{f/y})^{2-n}\left(\frac{y}{R}\right)^n. \tag{43}$$

Figure 2 shows that, as the non-Newtonian Prandtl number increases, the temperature profile becomes flatter in the turbulent core and steeper near the wall, until the latter region exerts the dominant influence on heat transfer, in accordance with the conclusions reached earlier, below Eq. (13).

All of these considerations underline the importance of assumptions about the wall region when N_{Pr}^+ is high. It seems likely that the level of agreement obtained with the simplest relationship (Eq. 22) is merely fortuitous, so that adjustment of y_L^+ from 5^n to a somewhat lower value, in order to obtain a closer fit to the data, would be illusory. Genuine improvement must await clarification of flow in the wall region, perhaps using laser techniques.

NOTATION

C_n	is defined by Eq. 23, dimensionless;
c_p	is the specific heat, BTU/lb · °R;
D	is the tube diameter, ft;
f	is the friction factor (Eq. 12), dimensionless;
g_c	is the conversion factor, 32.174 lb mass · ft/lb force · sec^2;
h	is the coefficient of heat transfer, BTU/h · ft^2 · °R;
K	is the fluid consistency index, lb mass · sec^{n-2} · ft^{-1};
k	is the thermal conductivity, BTU/sec · ft^2 · °R/ft, or BTU/h · ft^2 · °R/ft;
m	is the mass, lb mass;
n	is the flow behavior index, dimensionless;
N_{Nu}	is the Nusselt number, h D/k, dimensionless;
N_{Pr}^+	is a Prandtl number defined by Eqs. (3-4), dimensionless;
N_{Re}^+	is a Reynolds number defined by Eqs. (3-4), dimensionless;
N_{St}	is the Stanton number = $N_{Nu}/N_{Re}^+ N_{Pr}^+$, dimensionless;
Q_W	is defined below Eq. 39;
q, q_W	are heat flux and heat flux at the wall respectively, BTU/h · ft^2;
R	is the tube radius, ft;
r	is the radial distance, ft;
T	is the temperature, °R;

T^+, T_c, T_L, T_m, T_w, T_2	are dimensionless temperature (Eq. (7)), centerline temperature, temperature at $y = \delta_L$, bulk average temperature, wall temperature, and T at y_2^+ respectively, °R;
u, u^+, u_L, u_m	are local velocity, u/u^*, velocity at $y = \delta_L$, maximum or centerline velocity, all in the x direction, ft/sec;
u^*	is the friction velocity, $\sqrt{\tau_w g_c/\rho}$, ft/sec or ft/h;
V	is the mean velocity in the x direction, ft/sec;
x	is the distance in the direction of flow, ft;
y, y^+, y_L^+, y_2^+	are distance normal to surface, also y = R – r, ft, defined by Eq. (8), y^+ at $y = \delta_L$, y^+ at intersection of Eqs. (35) and (36), (y^+ = dimensionless);
α	is the constant in Eq. (12), dimensionless;
α_t	is the thermal diffusivity, $k/\rho c_p$, ft^2/h;
β	is the constant in Eq. (12), dimensionless;
γ_1	is K/C_n;
δ_L	is the thickness of laminar sublayer, ft;
ε_H, ε_M	are eddy thermal and momentum diffusivities, ft^2/hr;
ρ	is the density, lb mass/ft^3;
τ_w	is the shear stress at a conduit wall, lb force/ft^2;
τ_{rx}	is the shear stress in x direction on surface normal to r, lb force/ft^2;
Ω'	is $\varepsilon_H/\varepsilon_M$.

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